Structured sparsity through convex optimization

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Outline

- **Introduction: Sparse methods for machine learning**
  - Need for structured sparsity: Going beyond the ℓ_1-norm
- **Classical approaches to structured sparsity**
  - Linear combinations of ℓ_q-norms
- **Structured sparsity through submodular functions**
  - Relaxation of the penalization of supports
  - Unified algorithms and analysis
- **Extensions**

Sparsity in supervised machine learning

- Observed data $(x_i, y_i) \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \dots, n$
  - Response vector $y = (y_1, \dots, y_n)^\top \in \mathbb{R}^n$
  - Design matrix $X = (x_1, \dots, x_n)^\top \in \mathbb{R}^{n \times p}$
- Regularized empirical risk minimization:

$$\min_{w \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, w^\top x_i) + \lambda \Omega(w) = \boxed{\min_{w \in \mathbb{R}^p} \; L(y, Xw) + \lambda \Omega(w)}$$

- Norm Ω to promote sparsity
  - square loss + ℓ_1-norm $\Rightarrow$ basis pursuit in signal processing (Chen et al., 2001), Lasso in statistics/machine learning (Tibshirani, 1996)
  - Proxy for interpretability
  - Allow high-dimensional inference: $\boxed{\log p = O(n)}$

Sparsity in unsupervised machine learning

- **Multiple** responses/signals $y = (y^1, \dots, y^k) \in \mathbb{R}^{n \times k}$

$$\min_{w^1,\dots,w^k \in \mathbb{R}^p} \sum_{j=1}^{k} \Big\{ L(y^j, Xw^j) + \lambda \Omega(w^j) \Big\}$$

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- **Only responses are observed** $\Rightarrow$ **Dictionary learning**
  - Learn $X = (x^1, \dots, x^p) \in \mathbb{R}^{n \times p}$ such that $\forall j,\ \|x^j\|_2 \leqslant 1$

$$\min_{X=(x^1,\dots,x^p)} \min_{w^1,\dots,w^k \in \mathbb{R}^p} \sum_{j=1}^{k} \left\{ L(y^j, Xw^j) + \lambda \Omega(w^j) \right\}$$

  - Olshausen and Field (1997); Elad and Aharon (2006); Mairal et al. (2009a)

- **sparse PCA**: replace $\|x^j\|_2 \leqslant 1$ by $\Theta(x^j) \leqslant 1$

Sparsity in signal processing

- **Multiple** responses/signals $x = (x^1, \dots, x^k) \in \mathbb{R}^{n \times k}$

$$\min_{\alpha^1,\dots,\alpha^k \in \mathbb{R}^p} \sum_{j=1}^{k} \left\{ L(x^j, D\alpha^j) + \lambda\Omega(\alpha^j) \right\}$$

- **Only responses are observed** $\Rightarrow$ **Dictionary learning**
  - Learn $D = (d^1, \dots, d^p) \in \mathbb{R}^{n \times p}$ such that $\forall j,\ \|d^j\|_2 \leqslant 1$

$$\min_{D=(d^1,\dots,d^p)} \min_{\alpha^1,\dots,\alpha^k \in \mathbb{R}^p} \sum_{j=1}^{k} \left\{ L(x^j, D\alpha^j) + \lambda\Omega(\alpha^j) \right\}$$

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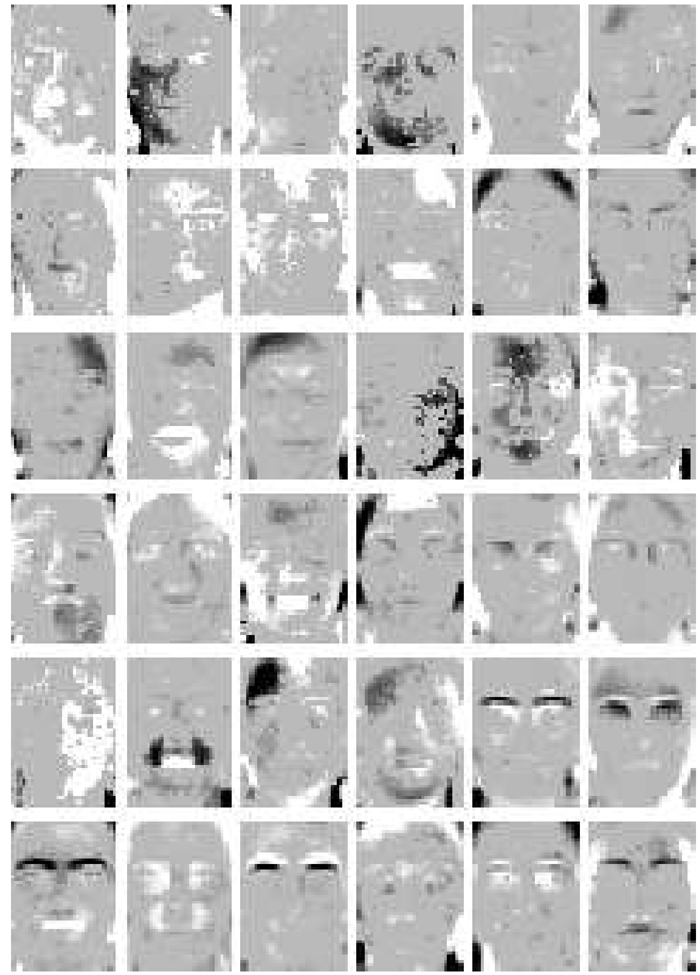
Why structured sparsity?

- **Interpretability**
  - Structured dictionary elements (Jenatton et al., 2009b)
  - Dictionary elements "organized" in a tree or a grid (Kavukcuoglu et al., 2009; Jenatton et al., 2010; Mairal et al., 2010)

Structured sparse PCA (Jenatton et al., 2009b)

raw data



sparse PCA

- Unstructed sparse PCA $\Rightarrow$ many zeros do not lead to better interpretability

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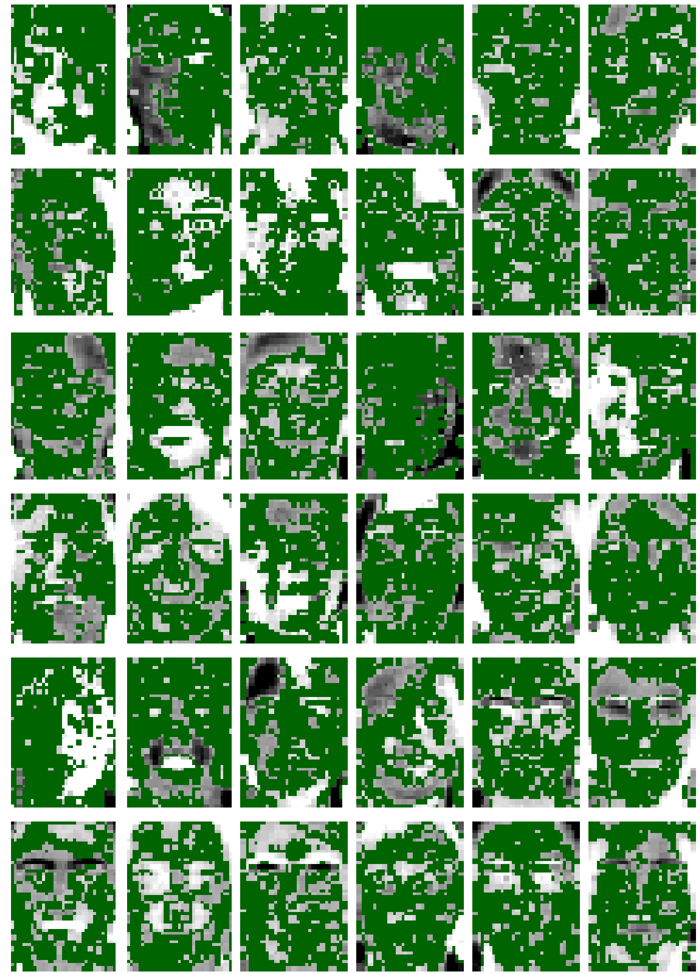

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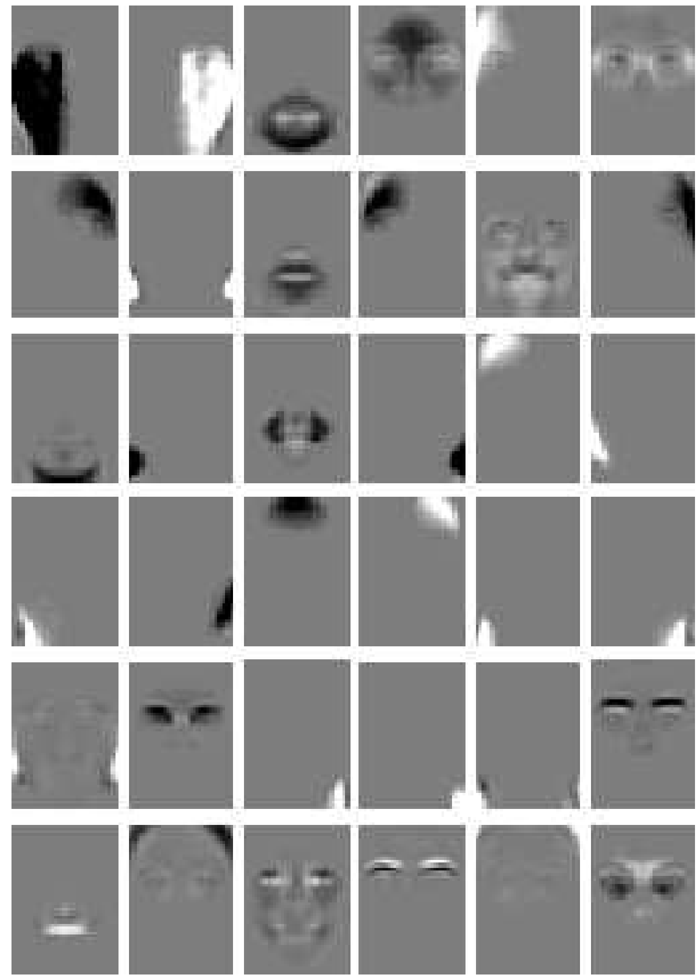

Structured sparse PCA

- Enforce selection of convex nonzero patterns ⇒ robustness to occlusion in face identification

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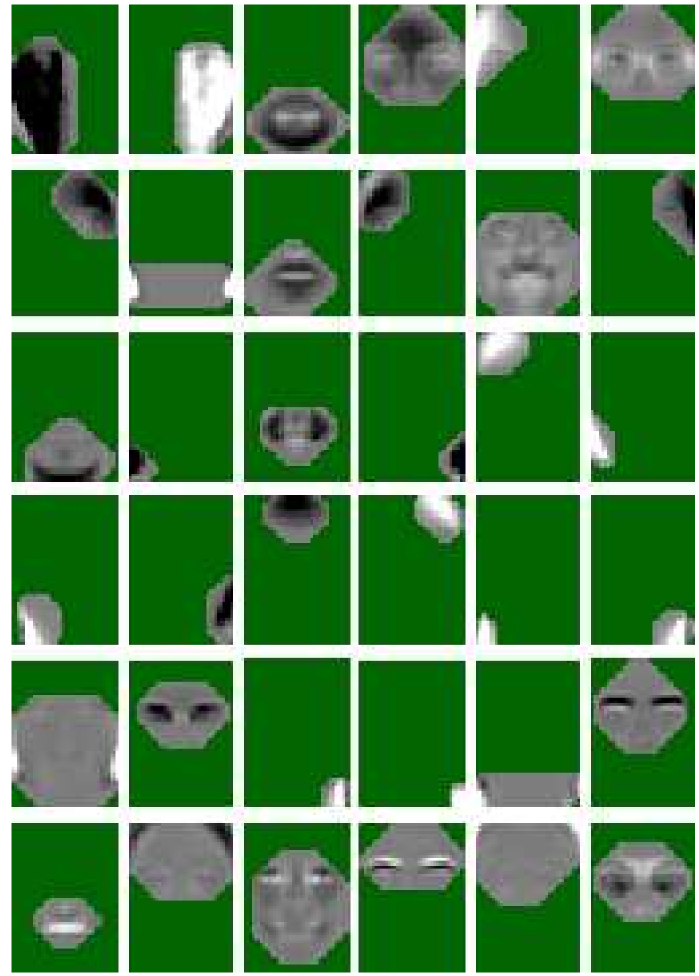

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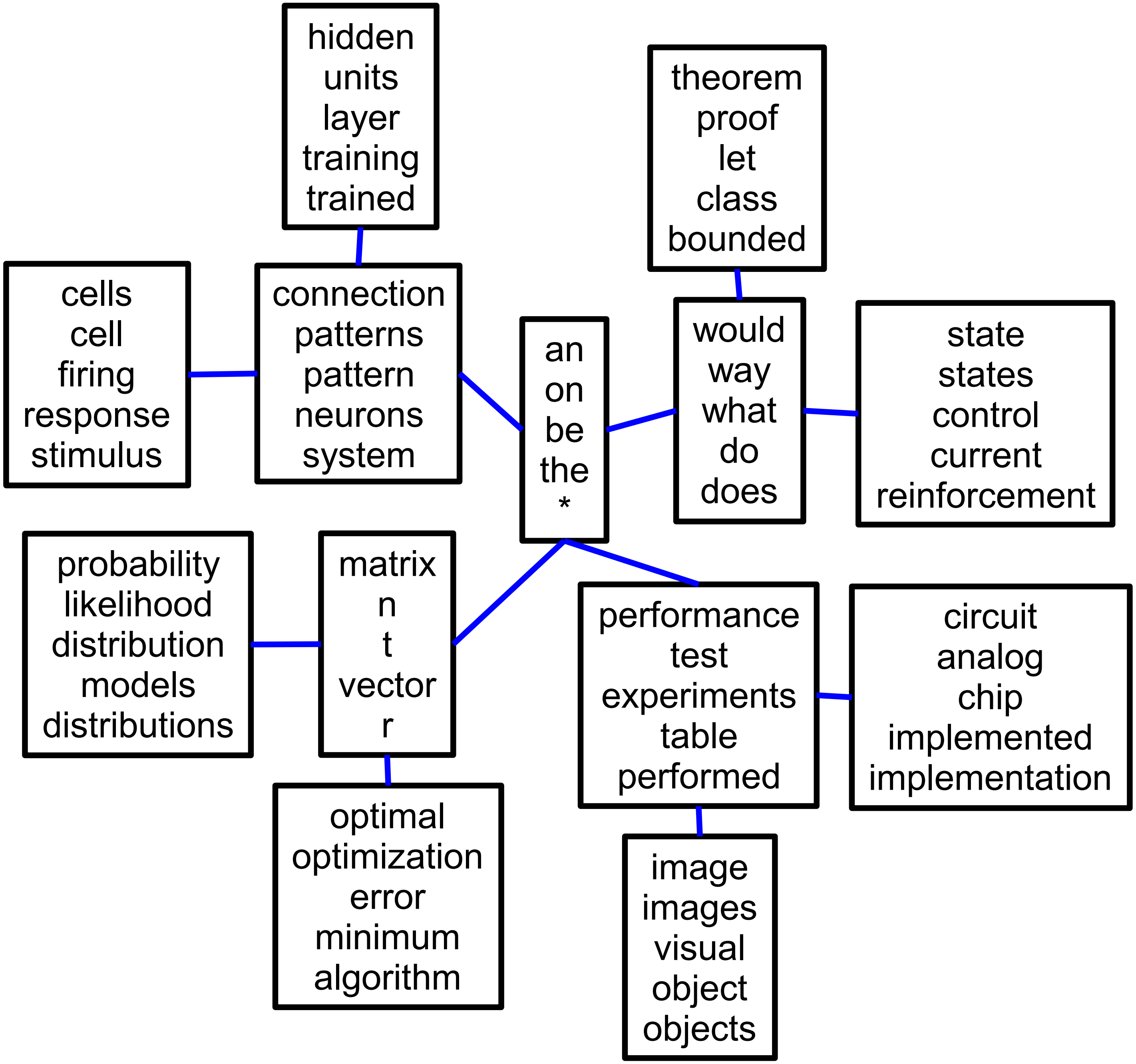
Modelling of text corpora (Jenatton et al., 2010)



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- **Stability and identifiability**
  - Optimization problem $\min_{w \in \mathbb{R}^p} L(y, Xw) + \lambda \|w\|_1$ is unstable
  - "Codes" w^j often used in later processing (Mairal et al., 2009c)
- **Prediction or estimation performance**
  - When prior knowledge matches data (Haupt and Nowak, 2006; Baraniuk et al., 2008; Jenatton et al., 2009a; Huang et al., 2009)
- **Numerical efficiency**
  - Non-linear variable selection with 2^p subsets (Bach, 2008)

Classical approaches to structured sparsity

- **Many application domains**
  - Computer vision (Cevher et al., 2008; Mairal et al., 2009b)
  - Neuro-imaging (Gramfort and Kowalski, 2009; Jenatton et al., 2011)
  - Bio-informatics (Rapaport et al., 2008; Kim and Xing, 2010)

- **Non-convex approaches**
  - Haupt and Nowak (2006); Baraniuk et al. (2008); Huang et al. (2009)

- **Convex approaches**
  - Design of sparsity-inducing norms

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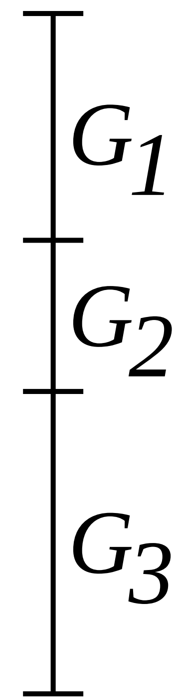
Sparsity-inducing norms

- **Popular choice for Ω**
  - The ℓ_1-ℓ_2 norm,

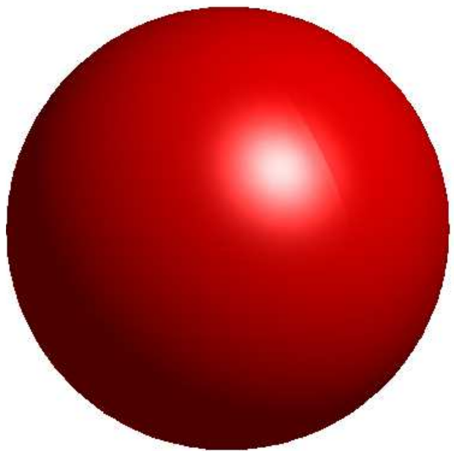
$$\sum_{G \in \mathbf{H}} \|w_G\|_2 = \sum_{G \in \mathbf{H}} \Big(\sum_{j \in G} w_j^2 \Big)^{1/2}$$



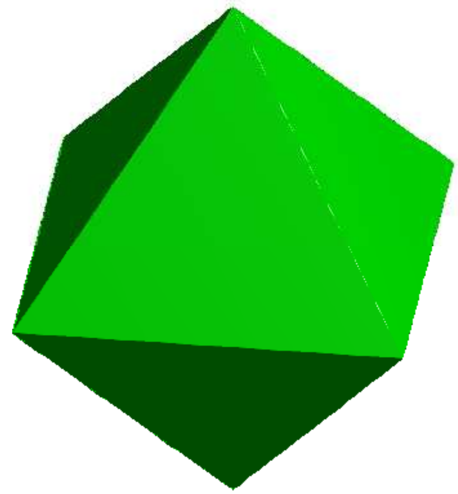
  - with $\mathbf{H}$ a partition of $\{1, \ldots, p\}$
  - The ℓ_1-ℓ_2 norm sets to zero groups of non-overlapping variables (as opposed to single variables for the ℓ_1-norm)
  - For the square loss, group Lasso (Yuan and Lin, 2006)

Unit norm balls

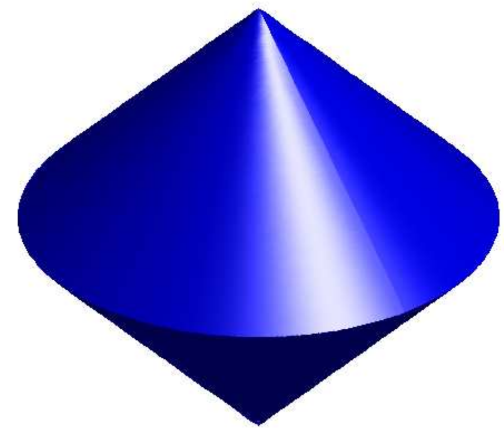
Geometric interpretation



$$\|w\|_2$$



$$\|w\|_1$$



$$\sqrt{w_1^2 + w_2^2} + |w_3|$$

Sparsity-inducing norms

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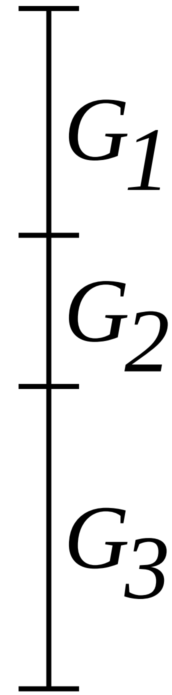

  - with $\mathbf{H}$ a partition of $\{1, \ldots, p\}$
  - The ℓ_1-ℓ_2 norm sets to zero groups of non-overlapping variables (as opposed to single variables for the ℓ_1-norm)
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- However, the ℓ_1-ℓ_2 norm encodes **fixed/static prior information**, requires to know in advance how to group the variables

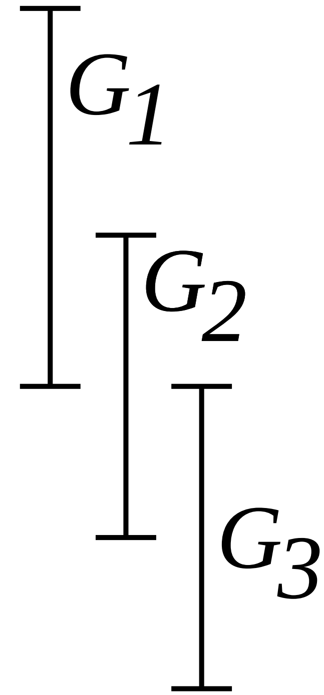
- What happens if the set of groups $\mathbf{H}$ is not a partition anymore?

Structured sparsity with overlapping groups (Jenatton, Audibert, and Bach, 2009a)

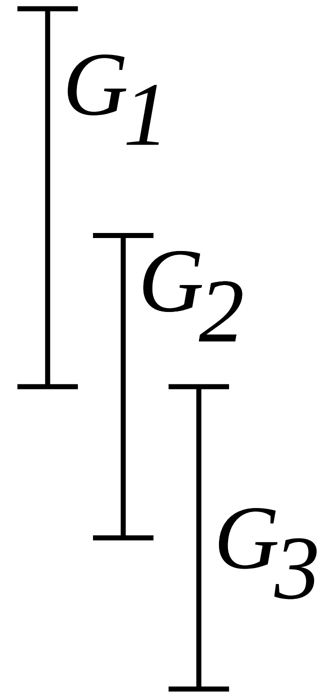
- When penalizing by the ℓ_1-ℓ_2 norm,

$$\sum_{G \in \mathbf{H}} \|w_G\|_2 = \sum_{G \in \mathbf{H}} \Big(\sum_{j \in G} w_j^2 \Big)^{1/2}$$

  - The ℓ_1 norm induces sparsity at the group level:
    - Some w_G's are set to zero
  - Inside the groups, the ℓ_2 norm does not promote sparsity



Structured sparsity with overlapping groups (Jenatton, Audibert, and Bach, 2009a)



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  - The ℓ_1 norm induces sparsity at the group level:
    - ∗ Some w_G's are set to zero
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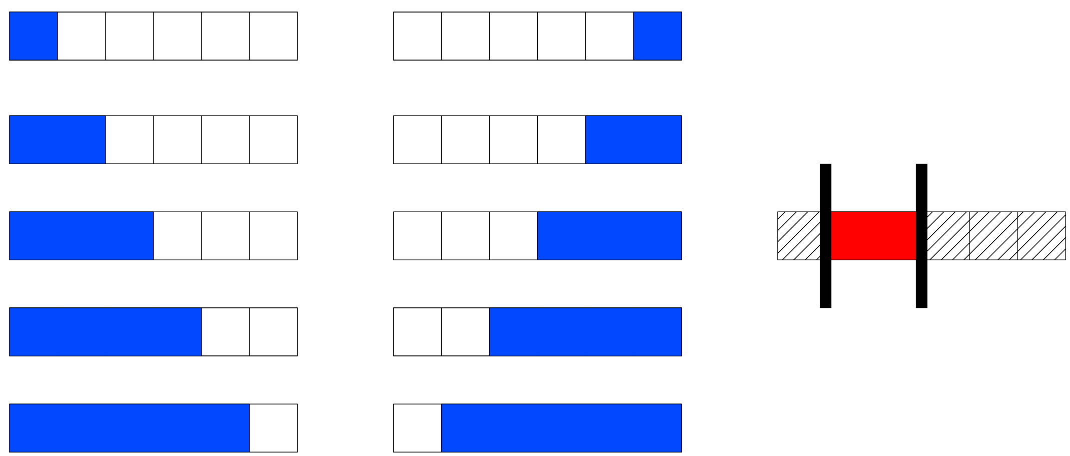
- The zero pattern of w is given by

$$\{j, \;\; w_j = 0\} = \bigcup_{G \in \mathbf{H}'} G \;\; \text{for some } \mathbf{H}' \subseteq \mathbf{H}$$

- **Zero patterns are unions of groups**

Examples of set of groups $\mathbb{H}$

- Selection of contiguous patterns on a sequence, $p = 6$

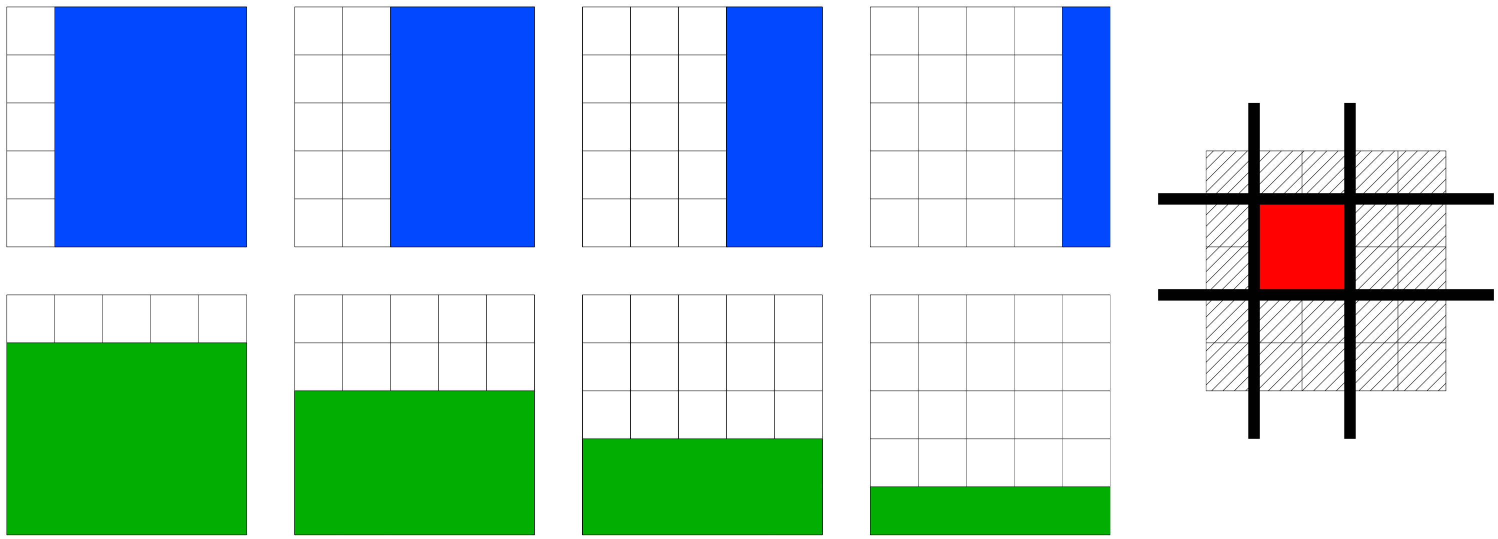


  – $\mathbf{H}$ is the set of blue groups

  – Any union of blue groups set to zero leads to the selection of a contiguous pattern

Examples of set of groups $\mathbb{H}$

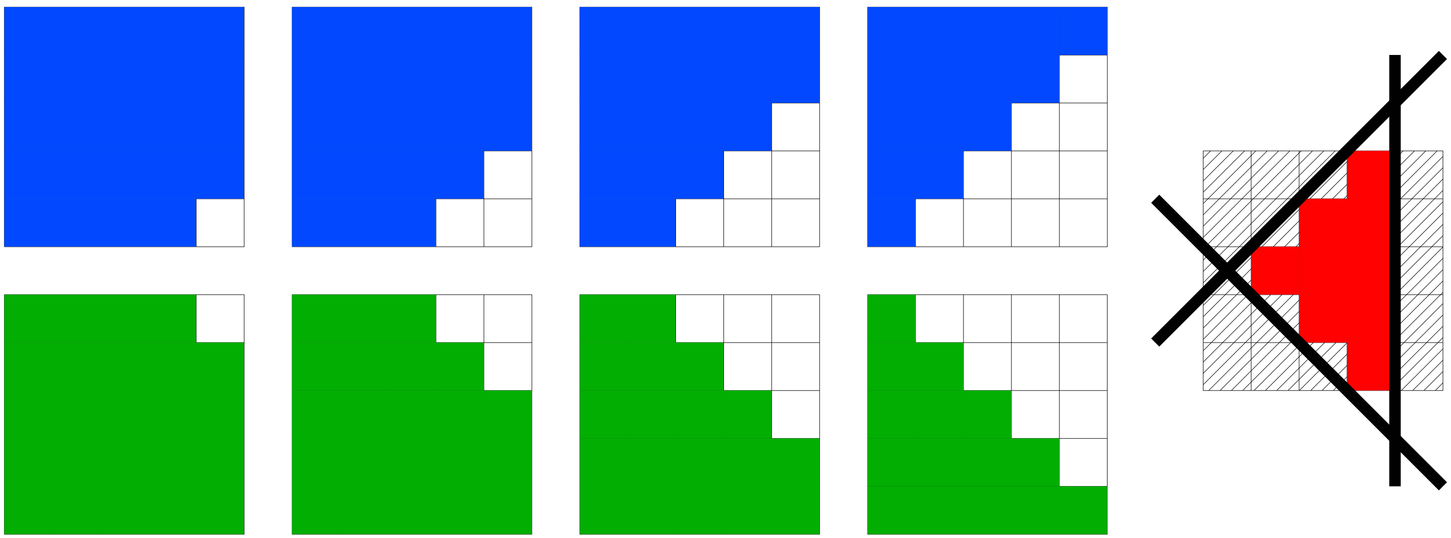
- Selection of rectangles on a 2-D grids, $p = 25$



  - $\mathbf{H}$ is the set of blue/green groups (with their not displayed complements)
  - Any union of blue/green groups set to zero leads to the selection of a rectangle

Examples of set of groups $\mathbb{H}$

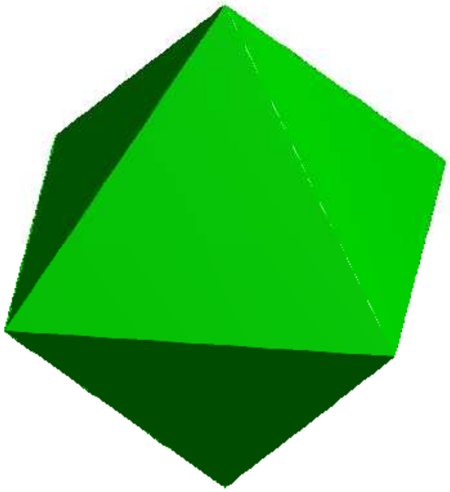
- Selection of diamond-shaped patterns on a 2-D grids, $p = 25$.



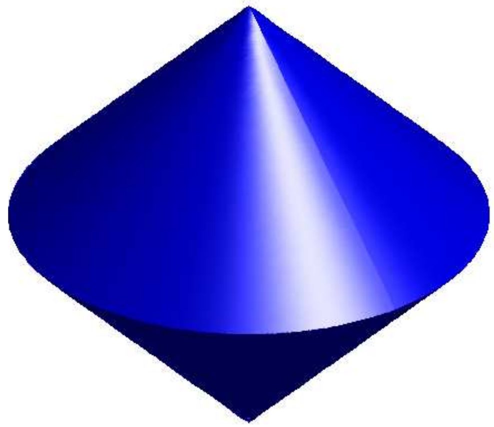
  - It is possible to extend such settings to 3-D space, or more complex topologies

Unit norm balls

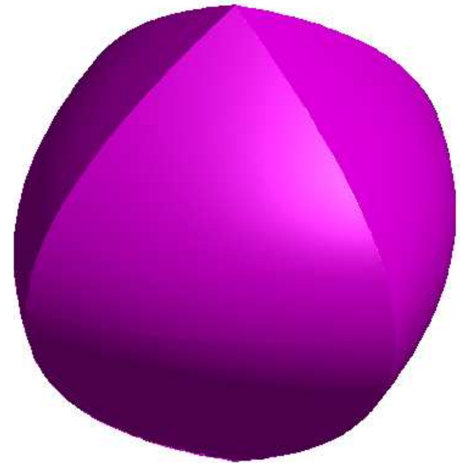
Geometric interpretation



$\|w\|_1$



$\sqrt{w_1^2 + w_2^2} + |w_3|$



$\|w\|_2 + |w_1| + |w_2|$

Optimization for sparsity-inducing norms
(see Bach, Jenatton, Mairal, and Obozinski, 2011)

- Gradient descent as a **proximal method** (differentiable functions)
  - $w_{t+1} = \arg\min_{w \in \mathbb{R}^p} J(w_t) + (w - w_t)^\top \nabla J(w_t) + \frac{L}{2}\|w - w_t\|_2^2$
  - $w_{t+1} = w_t - \frac{1}{L}\nabla J(w_t)$

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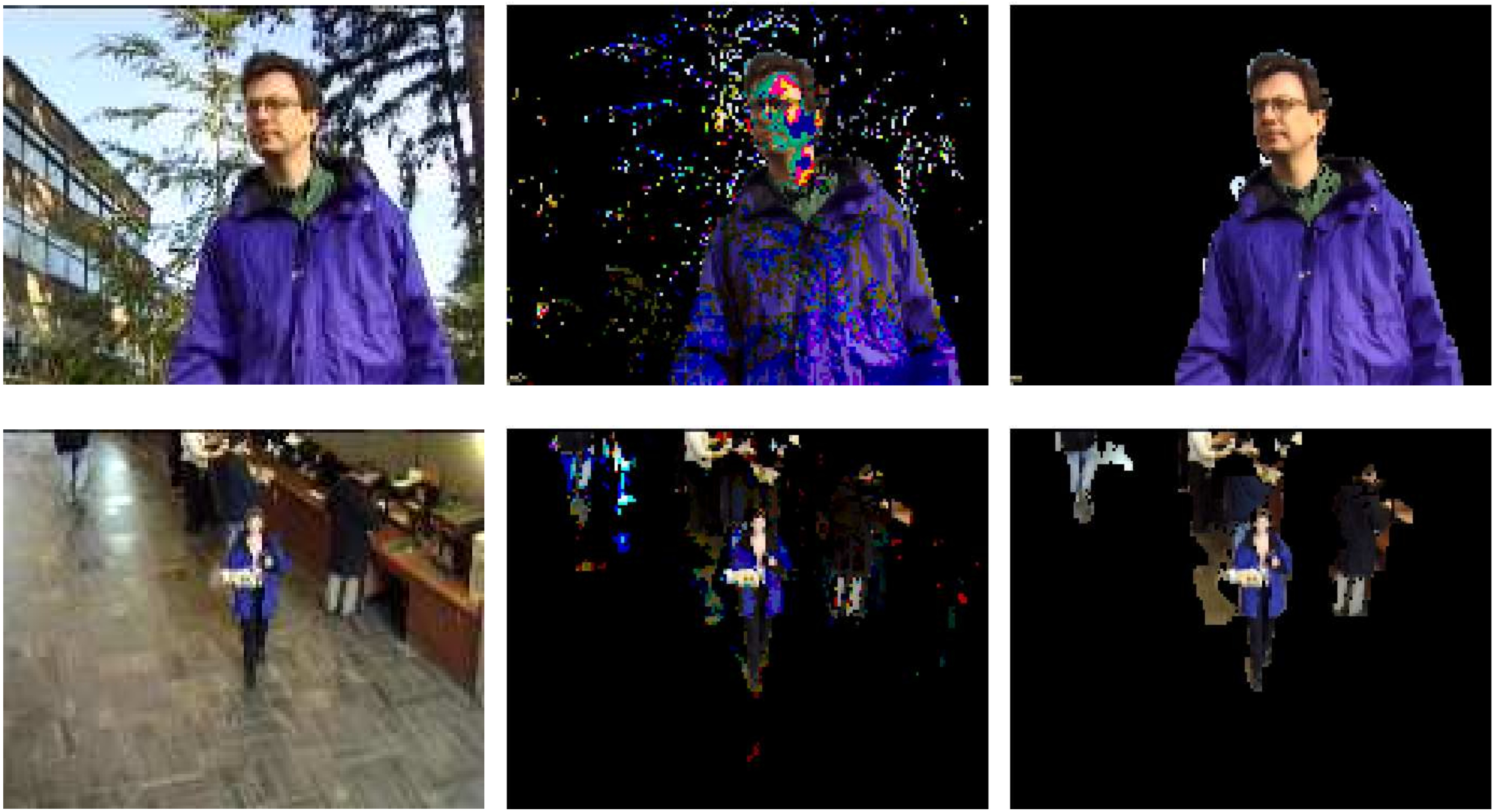
  - $w_{t+1} = w_t - \frac{1}{L}\nabla J(w_t)$
- Problems of the form: $\boxed{\min_{w \in \mathbb{R}^p} L(w) + \lambda\Omega(w)}$
  - $w_{t+1} = \arg\min_{w \in \mathbb{R}^p} L(w_t) + (w - w_t)^\top \nabla L(w_t) + \lambda\Omega(w) + \frac{L}{2}\|w - w_t\|_2^2$
  - $\Omega(w) = \|w\|_1 \Rightarrow$ **Thresholded gradient descent**
- Similar convergence rates than smooth optimization
  - Acceleration methods (Nesterov, 2007; Beck and Teboulle, 2009)

Application to background subtraction (Mairal, Jenatton, Obozinski, and Bach, 2010)

Input | ℓ_1-norm | Structured norm



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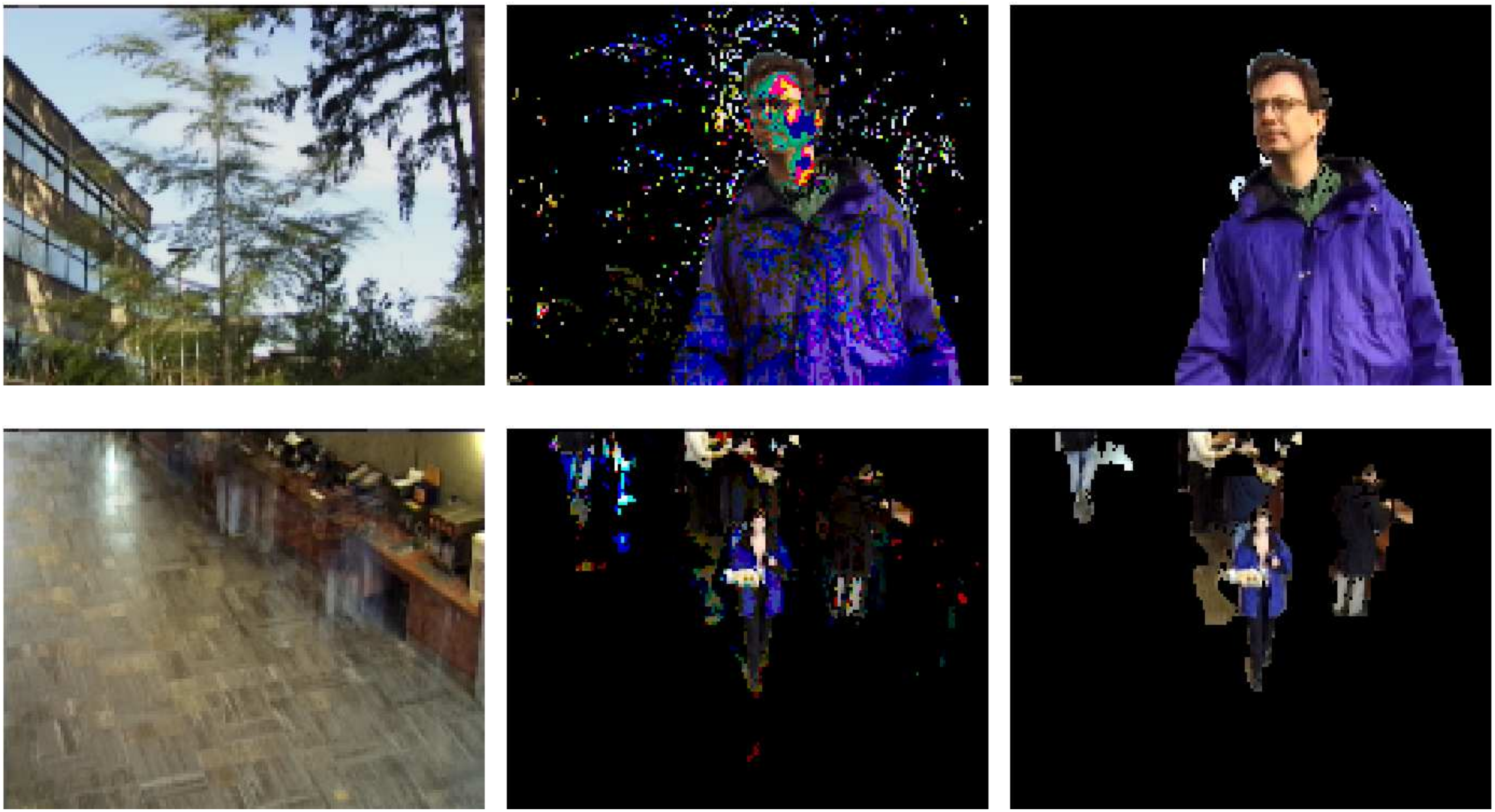
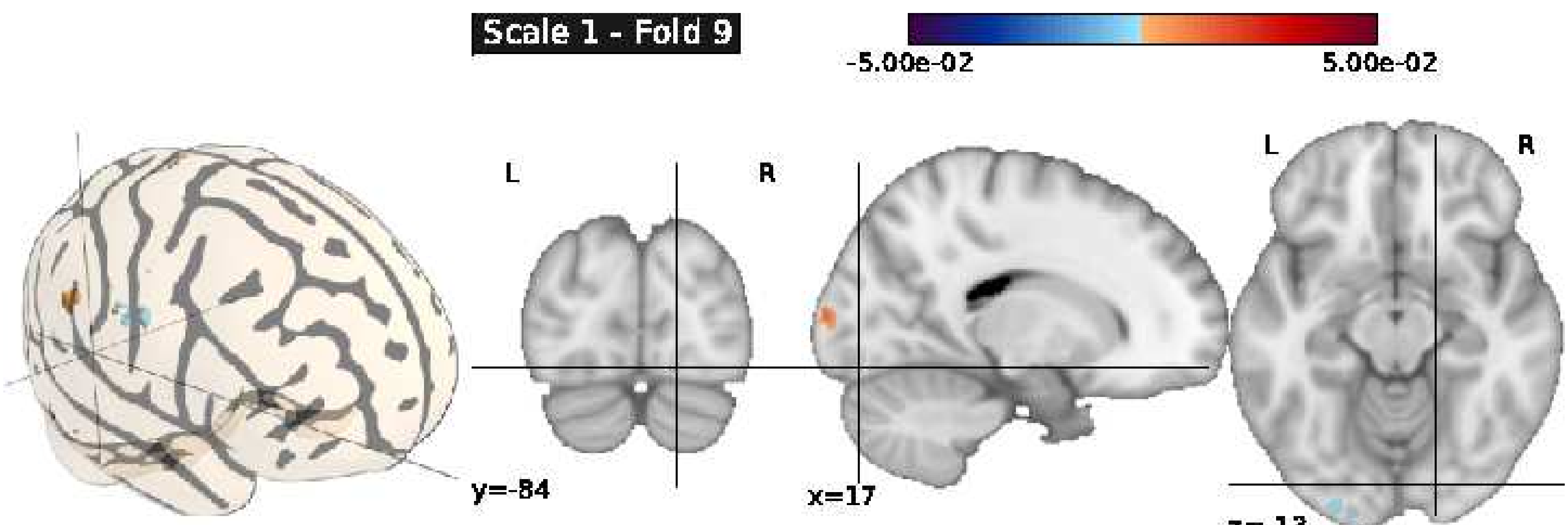
Background | ℓ_1-norm | Structured norm



Application to neuro-imaging

Structured sparsity for fMRI (Jenatton et al., 2011)

- “Brain reading”: prediction of (seen) object size
- Multi-scale activity levels through hierarchical penalization



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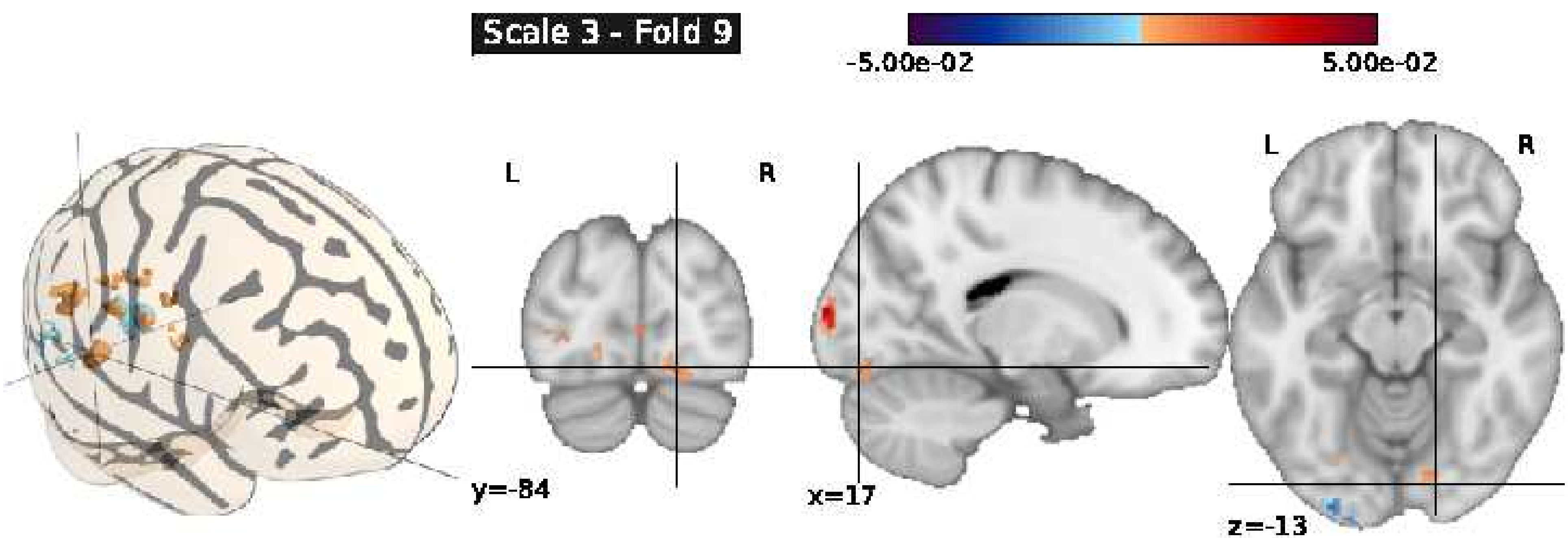

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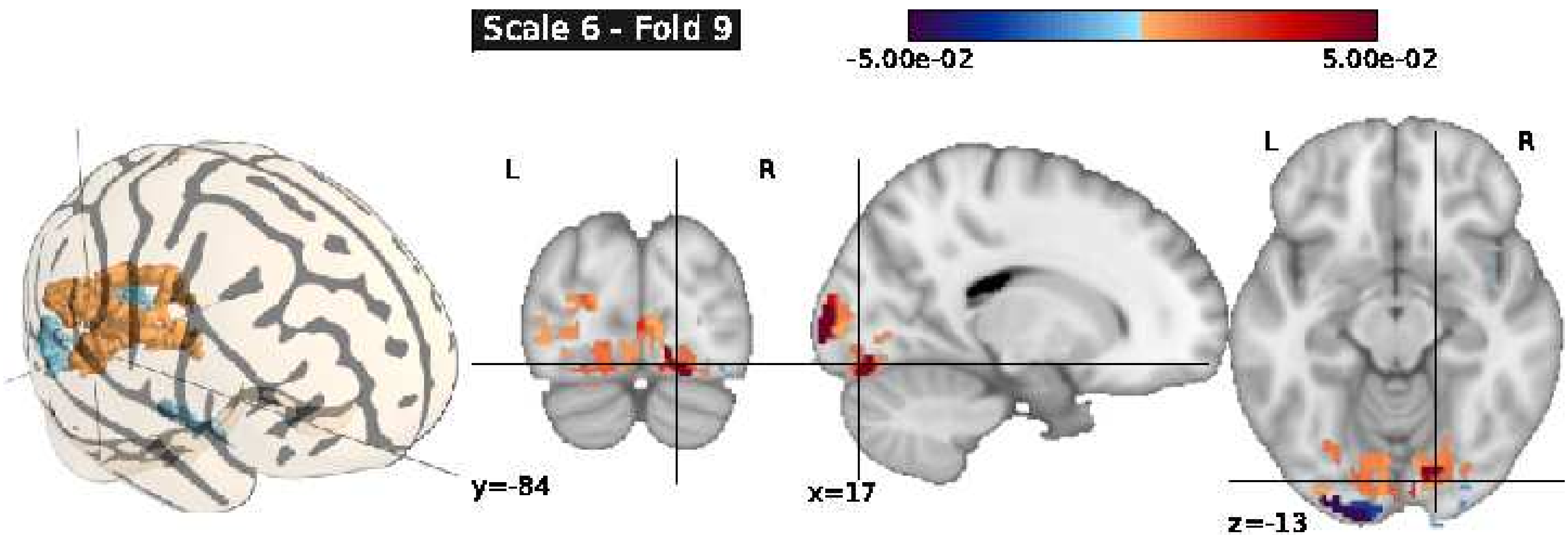


Sparse Structured PCA (Jenatton, Obozinski, and Bach, 2009b)

- Learning **sparse and structured dictionary elements**:

$$\min_{W\in\mathbb{R}^{k\times n},X\in\mathbb{R}^{p\times k}} \frac{1}{n}\sum_{i=1}^{n}\|y^i - Xw^i\|_2^2 + \lambda\sum_{j=1}^{p}\Omega(x^j) \text{ s.t. } \forall i,\ \|w^i\|_2 \leq 1$$

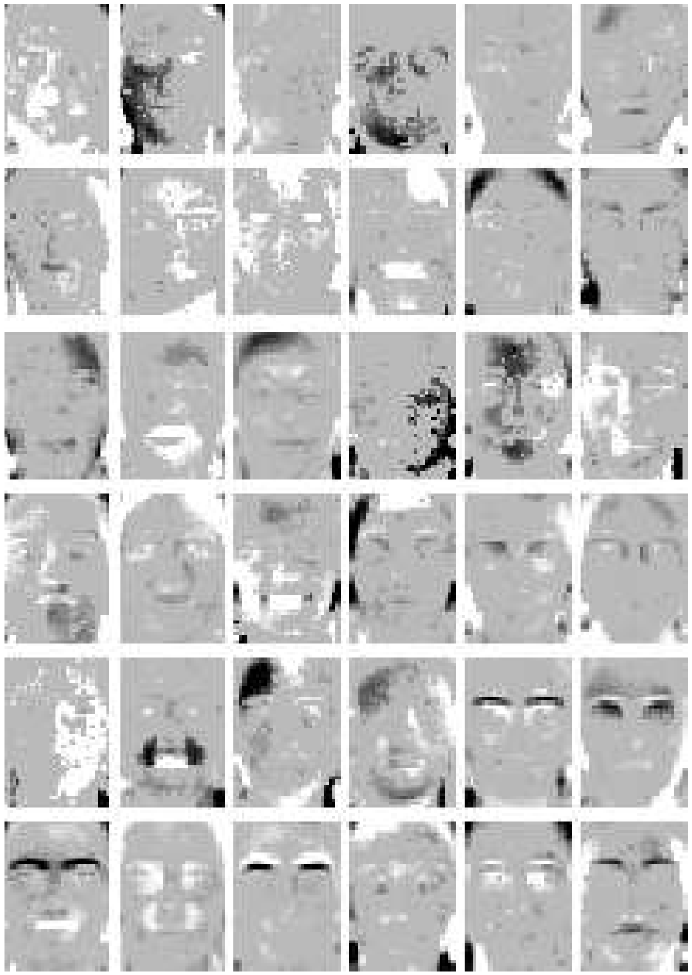
Application to face databases (1/3)

raw data

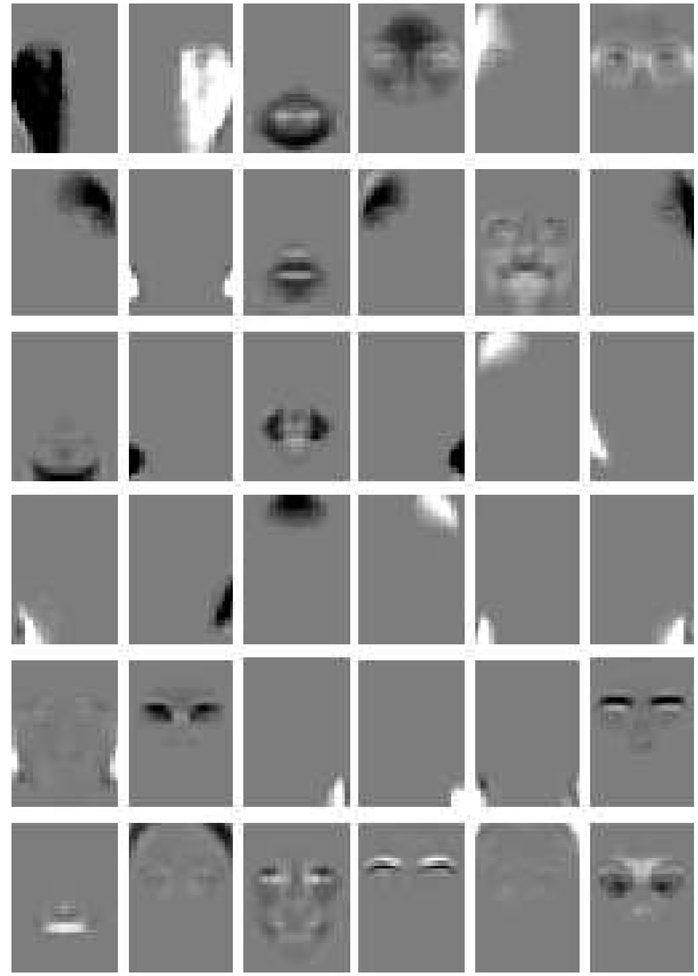
(unstructured) NMF

- NMF obtains partially local features

Application to face databases (2/3)



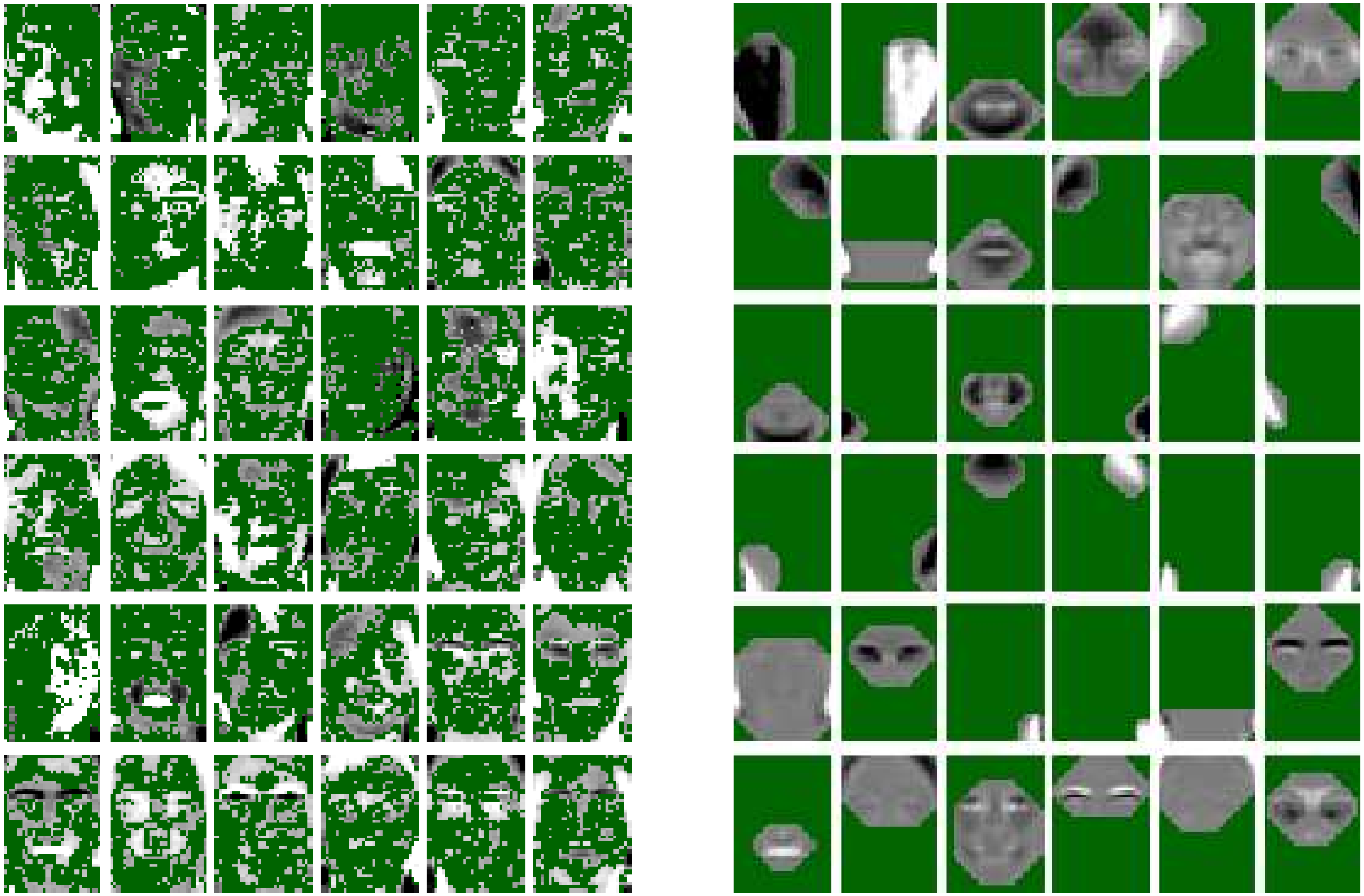
(unstructured) sparse PCA



Structured sparse PCA

- Enforce selection of convex nonzero patterns $\Rightarrow$ robustness to occlusion

Application to face databases (2/3)

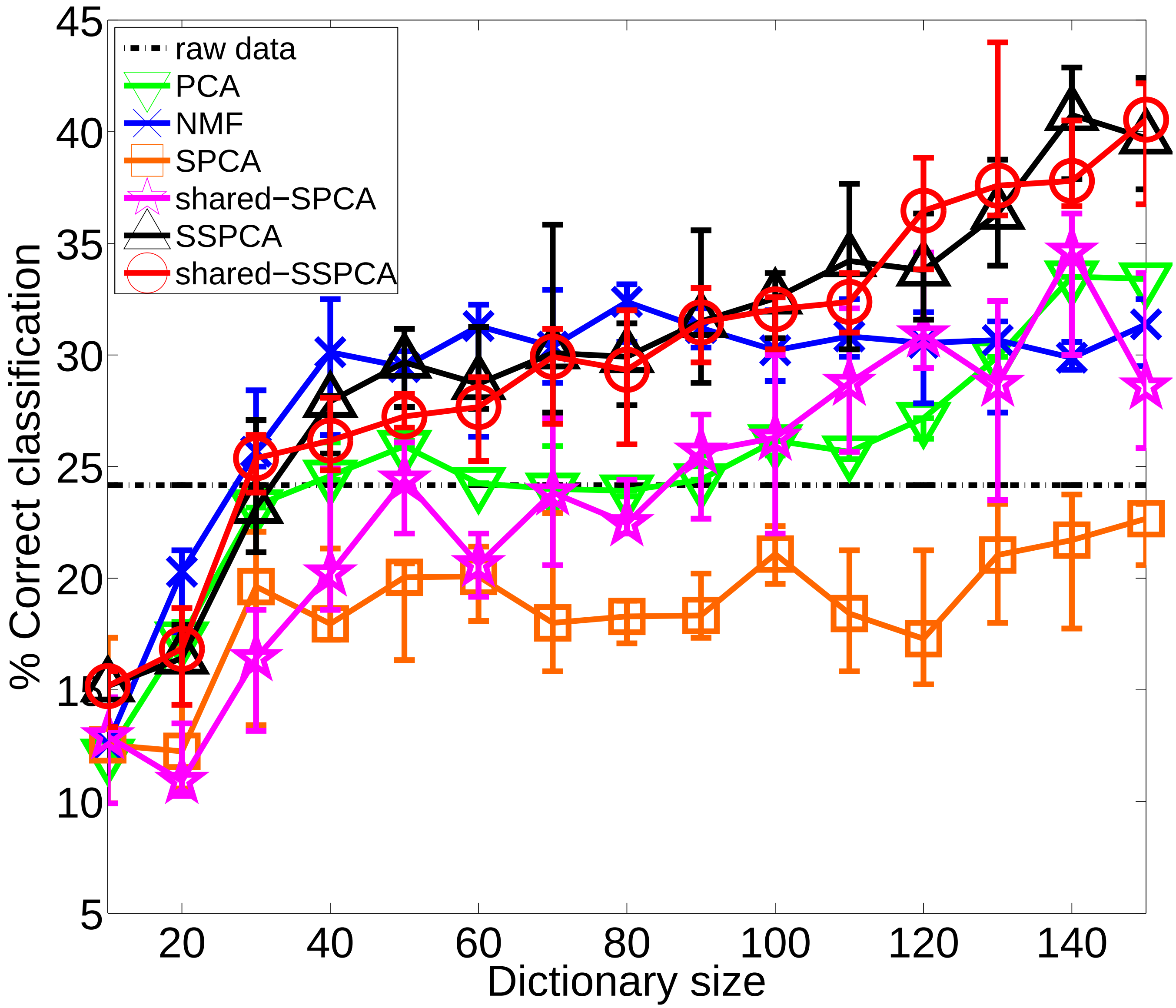


(unstructured) sparse PCA Structured sparse PCA

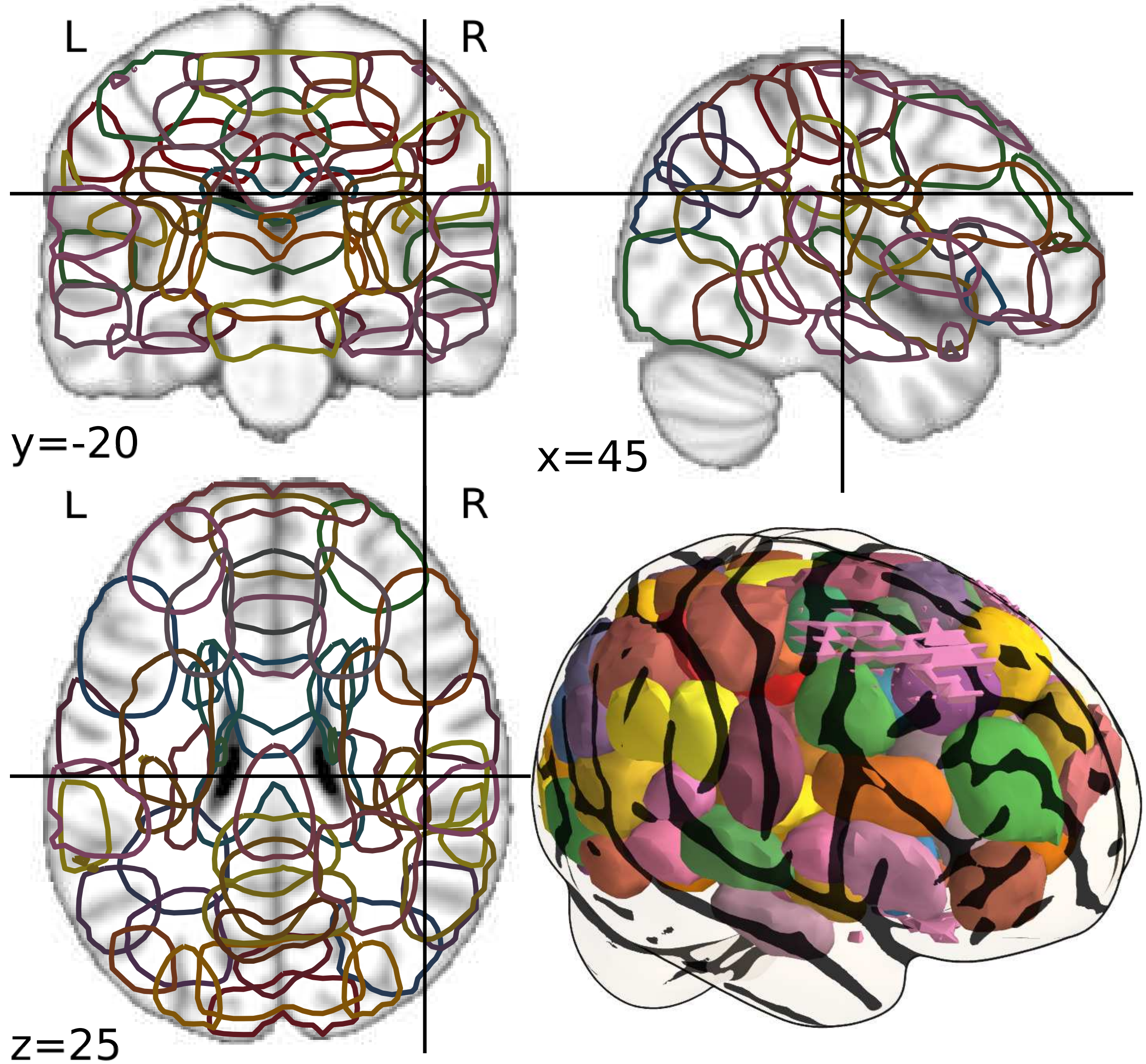
- Enforce selection of convex nonzero patterns $\Rightarrow$ robustness to occlusion

Application to face databases (3/3)

- Quantitative performance evaluation on classification task



Structured sparse PCA on resting state activity (Varoquaux, Jenatton, Gramfort, Obozinski, Thirion, and Bach, 2010)



Dictionary learning vs. sparse structured PCA
Exchange roles of X and w

- Sparse structured PCA (**structured dictionary elements**):

$$\min_{W\in\mathbb{R}^{k\times n}, X\in\mathbb{R}^{p\times k}} \frac{1}{n}\sum_{i=1}^{n}\|y^i - Xw^i\|_2^2 + \lambda\sum_{j=1}^{k}\Omega(x^j) \text{ s.t. } \forall i,\ \|w^i\|_2 \leq 1.$$

- Dictionary learning with **structured sparsity for codes** w:

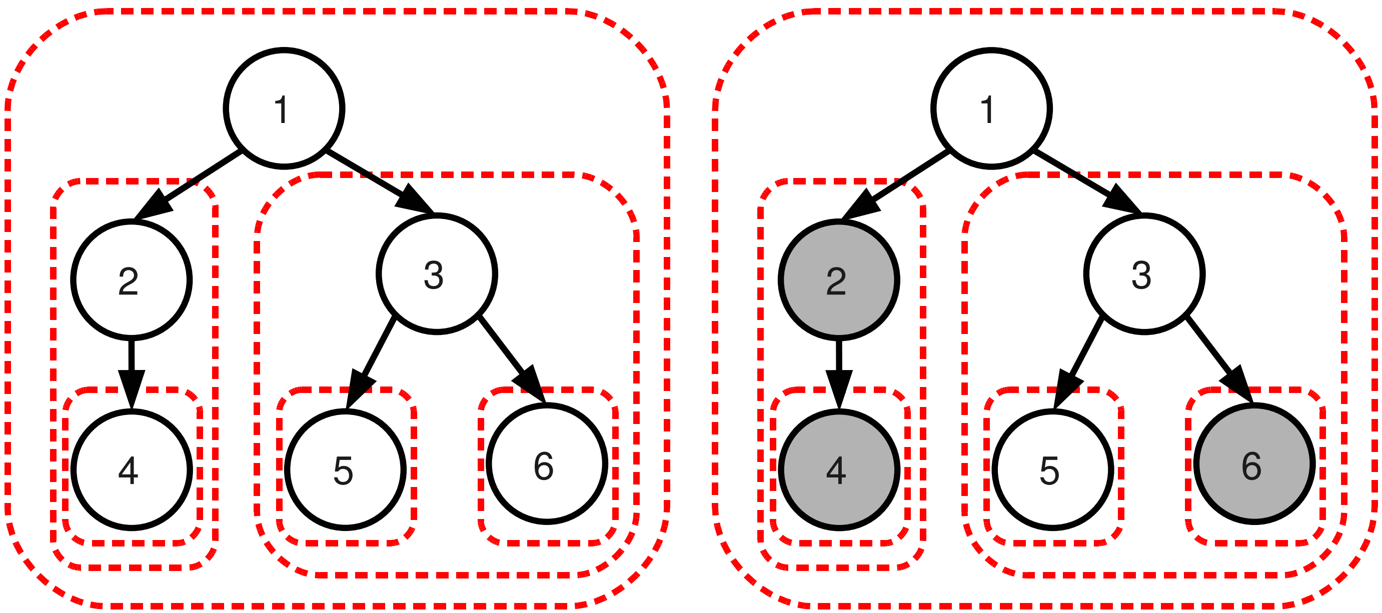
$$\min_{W\in\mathbb{R}^{k\times n}, X\in\mathbb{R}^{p\times k}} \frac{1}{n}\sum_{i=1}^{n}\|y^i - Xw^i\|_2^2 + \lambda\Omega(w^i) \text{ s.t. } \forall j,\ \|x^j\|_2 \leq 1.$$

- **Optimization**:
  - Alternating optimization
  - **Modularity of implementation** if proximal step is efficient (Jenatton et al., 2010; Mairal et al., 2010)

Hierarchical dictionary learning (Jenatton, Mairal, Obozinski, and Bach, 2010)

- Structure on codes w (not on dictionary X)

- Hierarchical penalization: $\Omega(w) = \sum_{G \in \mathbf{H}} \|w_G\|_2$ where groups G in $\mathbf{H}$ are equal to set of descendants of some nodes in a tree

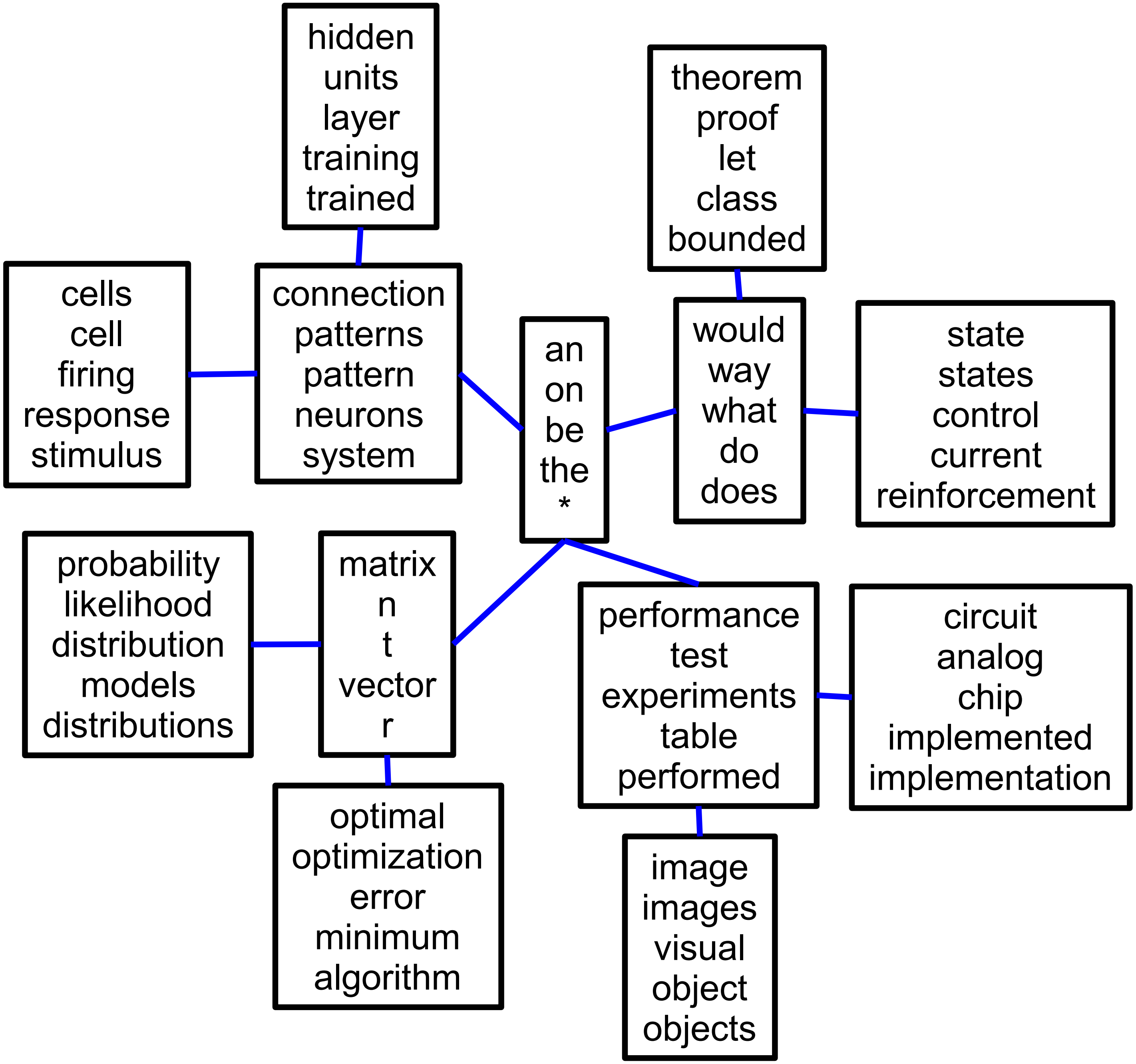


- Variable selected after its ancestors (Zhao et al., 2009; Bach, 2008)

Hierarchical dictionary learning
Modelling of text corpora

- Each document is modelled through word counts
- Low-rank matrix factorization of word-document matrix
- Probabilistic topic models (Blei et al., 2003)
  - Similar structures based on non parametric Bayesian methods (Blei et al., 2004)
  - **Can we achieve similar performance with simple matrix factorization formulation?**

Modelling of text corpora - Dictionary tree



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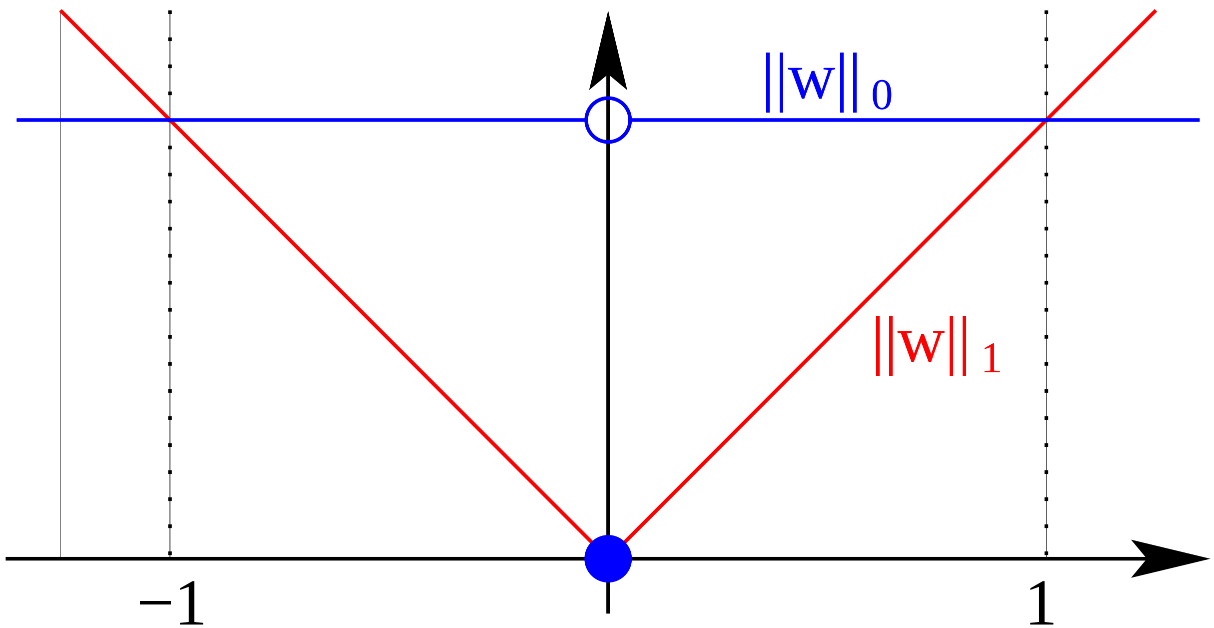
ℓ_1-norm = convex envelope of cardinality of support

- Let $w \in \mathbb{R}^p$. Let $V = \{1, \dots, p\}$ and $\mathrm{Supp}(w) = \{j \in V, \; w_j \neq 0\}$
- **Cardinality of support**: $\|w\|_0 = \mathrm{Card}(\mathrm{Supp}(w))$
- Convex envelope = largest convex lower bound (see, e.g., Boyd and Vandenberghe, 2004)



- ℓ_1-norm = convex envelope of ℓ_0-quasi-norm on the ℓ_∞-ball $[-1, 1]^p$

Convex envelopes of general functions of the support (Bach, 2010)

- Let $F : 2^V \to \mathbb{R}$ be a **set-function**
  - Assume F is **non-decreasing** (i.e., $A \subset B \Rightarrow F(A) \leqslant F(B)$)
  - Explicit prior knowledge on supports (Haupt and Nowak, 2006; Baraniuk et al., 2008; Huang et al., 2009)

- Define $\Theta(w) = F(\text{Supp}(w))$: How to get its convex envelope?
  1. Possible if F is also **submodular**
  2. Allows **unified** theory and algorithm
  3. Provides **new** regularizers

Submodular functions (Fujishige, 2005; Bach, 2010)

- $F : 2^V \to \mathbb{R}$ is **submodular** if and only if

$$\forall A, B \subset V, \quad F(A) + F(B) \geqslant F(A \cap B) + F(A \cup B)$$
$$\Leftrightarrow \quad \forall k \in V, \quad A \mapsto F(A \cup \{k\}) - F(A) \text{ is non-increasing}$$

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- Used in several areas of signal processing and machine learning
  - Total variation/graph cuts (Chambolle, 2005; Boykov et al., 2001)
  - Optimal design (Krause and Guestrin, 2005)

Submodular functions - Examples

- Concave functions of the cardinality: $g(|A|)$
- Cuts
- Entropies
  - $H((X_k)_{k \in A})$ from p random variables $X_1, \dots, X_p$
- Network flows
  - Efficient representation for set covers
- Rank functions of matroids

Submodular functions - Lovász extension

- Subsets may be identified with elements of $\{0,1\}^p$
- Given any set-function F and w such that $w_{j_1} \geqslant \cdots \geqslant w_{j_p}$, define:

$$f(w) = \sum_{k=1}^{p} w_{j_k}[F(\{j_1,\dots,j_k\}) - F(\{j_1,\dots,j_{k-1}\})]$$

  - If $w = 1_A$, $f(w) = F(A) \Rightarrow$ extension from $\{0,1\}^p$ to $\mathbb{R}^p$
  - f is piecewise affine and positively homogeneous

- F is submodular if and only if f is convex (Lovász, 1982)
  - Minimizing $f(w)$ on $w \in [0,1]^p$ equivalent to minimizing F on 2^V

Submodular functions and structured sparsity

- Let $F : 2^V \to \mathbb{R}$ be a **non-decreasing submodular set-function**
- **Proposition**: the convex envelope of $\Theta : w \mapsto F(\mathrm{Supp}(w))$ on the ℓ_∞-ball is $\Omega : w \mapsto f(|w|)$ where f is the Lovász extension of F

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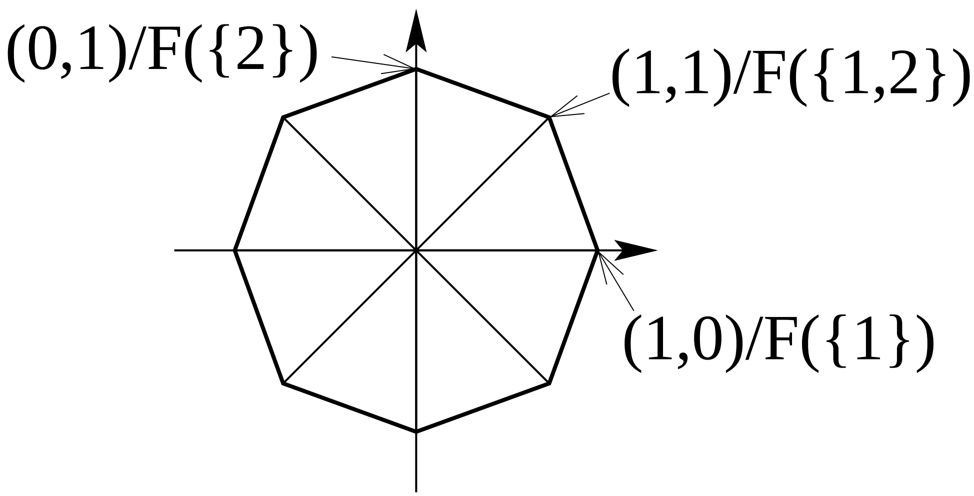
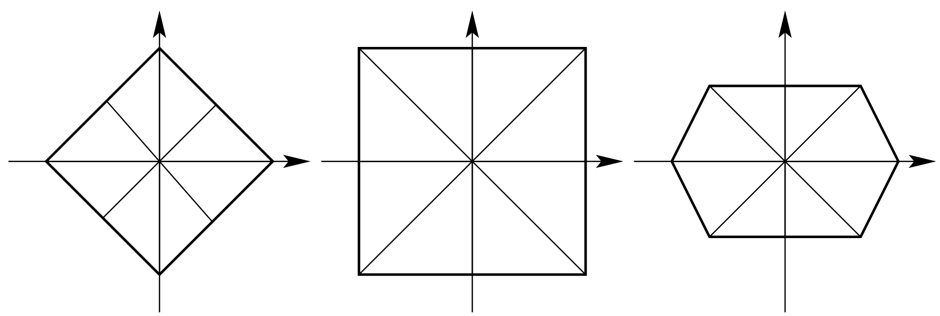
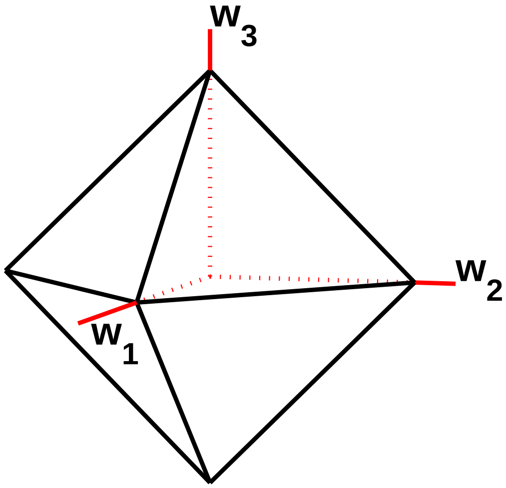
- **Sparsity-inducing properties**: Ω is a polyhedral norm



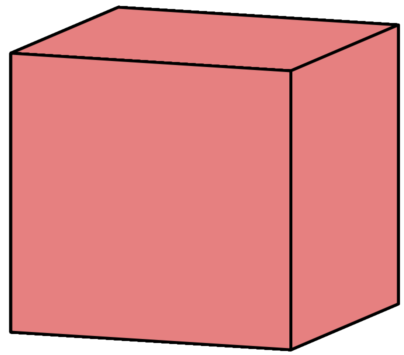


  – A if stable if for all $B \supset A$, $B \neq A \Rightarrow F(B) > F(A)$
  – With probability one, stable sets are the only allowed active sets

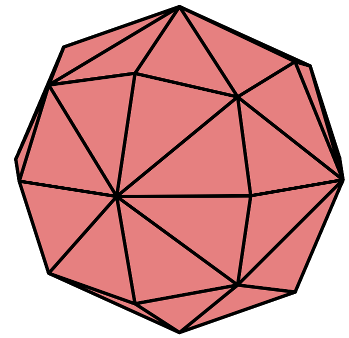
Polyhedral unit balls



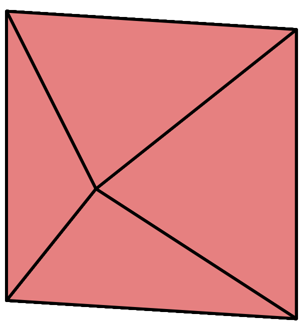
$F(A) = |A|$
$\Omega(w) = \|w\|_1$



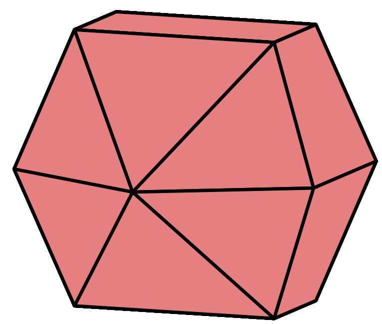
$F(A) = \min\{|A|, 1\}$
$\Omega(w) = \|w\|_\infty$



$F(A) = |A|^{1/2}$
all possible extreme points



$F(A) = 1_{\{A \cap \{1\} \neq \varnothing\}} + 1_{\{A \cap \{2,3\} \neq \varnothing\}}$
$\Omega(w) = |w_1| + \|w_{\{2,3\}}\|_\infty$



$F(A) = 1_{\{A \cap \{1,2,3\} \neq \varnothing\}}$
$+ 1_{\{A \cap \{2,3\} \neq \varnothing\}} + 1_{\{A \cap \{3\} \neq \varnothing\}}$
$\Omega(w) = \|w\|_\infty + \|w_{\{2,3\}}\|_\infty + |w_3|$

Conclusion

- **Structured sparsity for machine learning and statistics**
  - Many applications (image, audio, text, etc.)
  - May be achieved through structured sparsity-inducing norms
  - Link with submodular functions: unified analysis and algorithms

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  - Many applications (image, audio, text, etc.)
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- **On-going/related work on structured sparsity**
  - Norm design beyond submodular functions
  - Complementary approach of Jacob, Obozinski, and Vert (2009)
  - Links with greedy methods (Haupt and Nowak, 2006; Baraniuk et al., 2008; Huang et al., 2009)
  - Achieving $\log p = O(n)$ algorithmically (Bach, 2008)

References

F. Bach. Exploring large feature spaces with hierarchical multiple kernel learning. In *Advances in Neural Information Processing Systems*, 2008.

F. Bach. Structured sparsity-inducing norms through submodular functions. In *NIPS*, 2010.

F. Bach. Convex analysis and optimization with submodular functions: a tutorial. Technical Report 00527714, HAL, 2010.

F. Bach, R. Jenatton, J. Mairal, and G. Obozinski. Optimization with sparsity-inducing penalties. Technical Report 00613125, HAL, 2011.

R. G. Baraniuk, V. Cevher, M. F. Duarte, and C. Hegde. Model-based compressive sensing. Technical report, arXiv:0808.3572, 2008.

A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.

D. Blei, A. Ng, and M. Jordan. Latent dirichlet allocation. *The Journal of Machine Learning Research*, 3:993–1022, January 2003.

D. Blei, T.L. Griffiths, M.I. Jordan, and J.B. Tenenbaum. Hierarchical topic models and the nested Chinese restaurant process. *Advances in neural information processing systems*, 16:106, 2004.

S. P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Y. Boykov, O. Veksler, and R. Zabih. Fast approximate energy minimization via graph cuts. *IEEE Trans. PAMI*, 23(11):1222–1239, 2001.

V. Cevher, M. F. Duarte, C. Hegde, and R. G. Baraniuk. Sparse signal recovery using markov random fields. In *Advances in Neural Information Processing Systems*, 2008.

A. Chambolle. Total variation minimization and a class of binary MRF models. In *Energy Minimization Methods in Computer Vision and Pattern Recognition*, pages 136–152. Springer, 2005.

S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. *SIAM Review*, 43(1):129–159, 2001.

M. Elad and M. Aharon. Image denoising via sparse and redundant representations over learned dictionaries. *IEEE Transactions on Image Processing*, 15(12):3736–3745, 2006.

S. Fujishige. *Submodular Functions and Optimization*. Elsevier, 2005.

A. Gramfort and M. Kowalski. Improving M/EEG source localization with an inter-condition sparse prior. In *IEEE International Symposium on Biomedical Imaging*, 2009.

J. Haupt and R. Nowak. Signal reconstruction from noisy random projections. *IEEE Transactions on Information Theory*, 52(9):4036–4048, 2006.

J. Huang, T. Zhang, and D. Metaxas. Learning with structured sparsity. In *Proceedings of the 26th International Conference on Machine Learning (ICML)*, 2009.

L. Jacob, G. Obozinski, and J.-P. Vert. Group Lasso with overlaps and graph Lasso. In *Proceedings of the 26th International Conference on Machine Learning (ICML)*, 2009.

R. Jenatton, J.Y. Audibert, and F. Bach. Structured variable selection with sparsity-inducing norms. Technical report, arXiv:0904.3523, 2009a.

R. Jenatton, G. Obozinski, and F. Bach. Structured sparse principal component analysis. Technical report, arXiv:0909.1440, 2009b.

R. Jenatton, J. Mairal, G. Obozinski, and F. Bach. Proximal methods for sparse hierarchical dictionary learning. In *Submitted to ICML*, 2010.

R. Jenatton, A. Gramfort, V. Michel, G. Obozinski, E. Eger, F. Bach, and B. Thirion. Multi-scale mining of fmri data with hierarchical structured sparsity. Technical report, Preprint arXiv:1105.0363, 2011. In submission to SIAM Journal on Imaging Sciences.

K. Kavukcuoglu, M. Ranzato, R. Fergus, and Y. LeCun. Learning invariant features through topographic filter maps. In *Proceedings of CVPR*, 2009.

S. Kim and E. P. Xing. Tree-guided group Lasso for multi-task regression with structured sparsity. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2010.

A. Krause and C. Guestrin. Near-optimal nonmyopic value of information in graphical models. In *Proc. UAI*, 2005.

L. Lovász. Submodular functions and convexity. *Mathematical programming: the state of the art, Bonn*, pages 235–257, 1982.

J. Mairal, F. Bach, J. Ponce, and G. Sapiro. Online learning for matrix factorization and sparse coding. Technical report, arXiv:0908.0050, 2009a.

J. Mairal, F. Bach, J. Ponce, G. Sapiro, and A. Zisserman. Non-local sparse models for image restoration. In *Computer Vision, 2009 IEEE 12th International Conference on*, pages 2272–2279. IEEE, 2009b.

J. Mairal, F. Bach, J. Ponce, G. Sapiro, and A. Zisserman. Supervised dictionary learning. *Advances in Neural Information Processing Systems (NIPS)*, 21, 2009c.

J. Mairal, R. Jenatton, G. Obozinski, and F. Bach. Network flow algorithms for structured sparsity. In

NIPS, 2010.

Y. Nesterov. Gradient methods for minimizing composite objective function. *Center for Operations Research and Econometrics (CORE), Catholic University of Louvain, Tech. Rep*, 76, 2007.

B. A. Olshausen and D. J. Field. Sparse coding with an overcomplete basis set: A strategy employed by V1? *Vision Research*, 37:3311–3325, 1997.

F. Rapaport, E. Barillot, and J.-P. Vert. Classification of arrayCGH data using fused SVM. *Bioinformatics*, 24(13):i375–i382, Jul 2008.

R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of The Royal Statistical Society Series B*, 58(1):267–288, 1996.

G. Varoquaux, R. Jenatton, A. Gramfort, G. Obozinski, B. Thirion, and F. Bach. Sparse structured dictionary learning for brain resting-state activity modeling. In *NIPS Workshop on Practical Applications of Sparse Modeling: Open Issues and New Directions*, 2010.

M. Yuan and Y. Lin. Model selection and estimation in regression with grouped variables. *Journal of The Royal Statistical Society Series B*, 68(1):49–67, 2006.

P. Zhao, G. Rocha, and B. Yu. Grouped and hierarchical model selection through composite absolute penalties. *Annals of Statistics*, 37(6A):3468–3497, 2009.